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From gravitons to giants

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Abstract: We discuss exact quantization of gravitational fluctuations in the half-BPS sector around $\mathrm{AdS}_5\times\mathrm{S}^5$ background, using the dual super Yang-Mills theory. For this purpose we employ the recently developed techniques for exact bosonization of a finite number N of fermions in terms of N bosonic oscillators. An exact computation of the three-point correlation function of gravitons for finite N shows that they become strongly coupled at sufficiently high energies, with an interaction that grows exponentially in N. We show that even at such high energies a description of the bulk physics in terms of weakly interacting particles can be constructed. The single particle states providing such a description are created by our bosonic oscillators or equivalently these are the multi-graviton states corresponding to the so-called Schur polynomials. Both represent single giant graviton states in the bulk. Multi-particle states corresponding to multi-giant gravitons are, however, different, since interactions among our bosons vanish identically, while the Schur polynomials are weakly interacting at high enough energies.

Keywords: Gauge-gravity correspondence, Matrix Models, AdS-CFT Correspondence, Field Theories in Lower Dimensions.

Contents

1. Introduction

In a recent work [1] techniques have been developed for exact bosonization of a finite number N of nonrelativistic fermions. This has opened up new possibilities for exploring sectors of string theory non-perturbatively. One of the key motivations for this work was the study by Lin, Lunin and Maldacena [2] of a class of half-BPS type IIB geometries in asymptotically AdS spaces[1] and their connection with the semiclassical states of a free fermi system. Taken together with [1], the LLM work offers an excellent laboratory to explore non-perturbative aspects of quantum gravity in the above sector. Such a study was initiated[2] in [1] using the exact bosonization methods developed there.[3] In the present work we will explore these issues in greater detail.

[1] Various extensions of [2] have been made. 1/4-BPS excitations were considered in [3], 1/8-BPS excitations in [4] while bubbling geometries in AdS_3 were investigated by [5, 6]. 1/2-BPS excitations of $AdS_5 \times \mathbb{R}P^5$ were considered in [7]. Extension to multi-charge 1/2-BPS case has been considered in [8].

[2] Related aspects of this issue were discussed in [9, 10]. See also [11, 12].

[3] This bosonization works for arbitrary fermion Hamiltonian and can also be applied to $c = 1$, free fermions on a circle (the Tomonaga problem), etc. Also see the remarks at the end of section 2.

JHEP03(2006)031

In the boundary super Yang-Mills theory, the states corresponding to half-BPS LLM geometries are described by N free fermions in a harmonic potential [13 – 15] (see also [16]). At large N, there is a semiclassical description of the states of this system in terms of droplets of fermi fluid in phase space; LLM showed that there is a similar structure in the classical geometries in the bulk. This semiclassical correspondence is already remarkable in the sense that it exhibits a noncommutative structure of two of the space coordinates [1, 2, 17]; however, finite N effects, corresponding to fully quantum mechanical aspects of bulk gravity, open up more interesting questions [1, 18]. Semiclassically, small fluctuations of the droplet boundaries correspond to small gravitational fluctuations around the classical geometries, as anticipated by [2] and shown by [19, 20] (in part using the symplectic form calculated by [21]). At finite N only those fluctuations of the fermi system which have low enough excitation energy compared to N can be identified with gravity modes propagating in the bulk. From the bulk gravity point of view, the relevant (dimensionless) scale is $R/l_p \sim N^{1/4}$, where R is the AdS radius and l_p is the ten-dimensional Planck scale, since beyond these energies perturbative corrections may be expected to become large. However, from an exact calculation in the boundary theory we find that perturbation theory actually breaks down much later, at energies of order $N^{1/2}$. It is possible that the reason for this is cancellations due to the high degree of supersymmetry in the half-BPS sector. A different argument for the existence of an energy scale of order $N^{1/2}$ exists [22] that suggests breakdown of weakly coupled gravity picture for the LLM gravitons at this scale. Essentially the argument is that the size of the wavefunction (in AdS_5) of an LLM graviton excitation decreases with energy and at an energy scale of order $N^{1/2}$ it becomes order Planck scale.

At still higher energies, one would expect a description of the bulk physics in terms of graviton excitations to break down. In fact, as we will see here, at sufficiently high energies gravitons cease to make sense as weakly coupled degrees of freedom since their correlations grow exponentially with N. This happens long before graviton energies are of order N. At sufficiently high energies, therefore, we need to seek out new weakly coupled degrees of freedom which can provide a more meaningful description of bulk physics than gravitons. We will explicitly find these degrees of freedom in this paper. As we will argue in this paper, there is strong evidence that these new degrees of freedom are giant gravitons [23 – 25]. In the boundary theory, the corresponding single-particle states are created by the oscillators of the bosonized theory. Since these are strictly non-interacting, one can obviously describe physics in terms of these at all energies. Remarkably, we find that the single-particle states created by the oscillators are also exactly the single-particle states corresponding to the combinations of multi-graviton states (for totally antisymmetric representaions) known as Schur polynomials. This does not hold for multi-particle states. In fact, there are small but non-zero correlations among the Schur polynomials at high energies, which distinguishes them from the oscillator degrees of freedom. In any case, giant gravitons, which are closely related to both these boundary degrees of freedom, have the right properties to provide a good description of the bulk physics at high energies. This transition from low-energy graviton degrees of freedom to more microscopic degrees of freedom at high energies is expected to happen in any consistent theory

JHEP03(2006)031

of gravity. The remarkable thing about the LLM system is that it provides us with a laboratory in which we can actually see this happening in a very controlled and explicit fashion.

The organization of this paper is as follows. In section 2 we will summarize the work of [1] [4] on the exact bosonization of a finite number N of nonrelativistic fermions. The presentation here is somewhat different and simpler. When applied to the LLM sector, we find that the bosonized theory is described by a free hamiltonian for N bosonic oscillators. In section 3 we will discuss correlation functions of the modes of fermion spatial density, which correspond to single trace operators in the boundary super Yang-Mills theory. These are the modes for small fluctuations of the bulk metric around $\mathrm{AdS}_5\times\ \mathrm{S}^5$ which preserve the half-BPS condition. We argue that the effective low-energy physics of these modes is described by a field theory with a small cubic coupling of $\mathrm{O}(1/N)$. However, the large-N perturbation theory breaks down when graviton energies are of order $\sqrt{N}$. This is shown by doing an exact calculation of graviton three-point function in section 3. This calculation also shows that gravitons cease to provide a meaningful description of the bulk physics at much higher energies, which may still be only a small fraction of N. Instead, at these energies we must use the giant gravitons to provide a meaningful description of the bulk physics. In section 3.5 we show that the single-particle states created by our bosonic oscillators are identical to the single-particle states created by the Schur polynomials for totally antisymmetric representaions. This is done by establishing a general relation between multitrace operators and the bosonic oscillators acting on the fermi vacuum. We end with a summary and some comments in section 4.

2. Review of exact bosonization

In this section we will review the techniques developed in [1] for an exact operator bosonization of a finite number of nonrelativistic fermions. The discussion here is somewhat different from that in [1]. Here, we will derive the first bosonization of [1] using somewhat simpler arguments, considerably simplifying the presentation and the formulae in the process. Moreover, the present derivation of bosonization rules is more intuitive, making its applications technically easier.

Consider a system of N fermions each of which can occupy a state in an infinite-dimensional Hilbert space $\mathcal{H}_f$. Suppose there is a countable basis of $\mathcal{H}_f$: $\{|m\rangle, m = 0, 1, \cdots, \infty\}$. For example, this could be the eigenbasis of a single-particle hamiltonian, $\hat{h}|m\rangle = \mathcal{E}(m)|m\rangle$, although other choices of basis would do equally well, as long as it is a countable basis. In the second quantized notation we introduce creation (annihilation) operators $\psi_m^\dagger$ (ψ_m) which create (destroy) particles in the state $|m\rangle$. These satisfy the anticommutation relations

$$\{\psi_m, \psi_n^\dagger\} = \delta_{mn}\,. \tag{2.1}$$

[4]The work in this paper discusses two different exact bosonizations of the fermi system; here we will limit our discussion to bosonization of the first type.

The N-fermion states are given by (linear combinations of)

$$|f_1, \cdots, f_N\rangle = \psi^\dagger_{f_1}\psi^\dagger_{f_2}\cdots\psi^\dagger_{f_N}|0\rangle_F \tag{2.2}$$

where f_m are arbitrary integers satisfying $0 \le f_1 < f_2 < \cdots < f_N$, and $|0\rangle_F$ is the usual Fock vacuum annihilated by $\psi_m, m = 0, 1, \cdots, \infty$.

It is clear that one can span the entire space of N-fermion states, starting from a given state $|f_1, \cdots, f_N\rangle$, by repeated application of the fermion bilinear operators

$$\Phi_{mn} = \psi^\dagger_m\,\psi_n\,. \tag{2.3}$$

However, the problem with these bosonic operators is that they are not independent; this is reflected in the W_∞ algebra that they satisfy,

$$[\Phi_{mn}, \Phi_{m'n'}] = \delta_{m'n}\Phi_{mn'} - \delta_{mn'}\Phi_{m'n}. \tag{2.4}$$

This is the operator version of the noncommutative constraint $u * u = u$ that the Wigner distribution u satisfies in the exact path-integral bosonization carried out in [26].

A new set of unconstrained bosonic operators was introduced in [1], N of them for N fermions. In effect, this set of bosonic operators provides the independent degrees of freedom in terms of which the above constraint is solved. Let us denote these operators by $\sigma_k,\ k = 1, 2, \cdots, N$ and their conjugates, $\sigma^\dagger_k,\ k = 1, 2, \cdots, N$. As we shall see shortly, these operators will turn out to be identical to the σ_k's used in [1]. The action of $\sigma^\dagger_k$ on a given fermion state $|f_1, \cdots, f_N\rangle$ is rather simple. It just takes each of the fermions in the top k occupied levels up by one step, as illustrated in figure 1. One starts from the fermion in the topmost occupied level, f_N, and moves it up by one step to $(f_N + 1)$, then the one below it up by one step, etc proceeding in this order, all the way down to the kth fermion from top, which is occupying the level f_{N-k+1} and is taken to the level $(f_{N-k+1} + 1)$. For the conjugate operation, σ_k, one takes fermions in the top occupied k levels down by one step, reversing the order of the moves. Thus, one starts by moving the fermion at the level f_{N-k+1} to the next level below at $(f_{N-k+1} - 1)$, and so on. Clearly, in this case the answer is nonzero only if the $(k+1)$th fermion from the top is not occupying the level immediately below the kth fermion , i.e. if $(f_{N-k+1} - f_{N-k} - 1) > 0$. If $k = N$ this condition must be replaced by $f_1 > 0$.

These operations are necessary and sufficient to move to any desired fermion state starting from a given state. This can be argued as follows. First, consider the following operator, $\sigma_{k-1}\,\sigma^\dagger_k$. Acting on an arbitrary fermion state, the first factor takes top k fermions up by one level; this is followed by bringing the top $(k-1)$ fermions down by one level. The net effect is that only the kth fermion from top is moved up by one level. In other words, $\sigma_{k-1}\ \sigma^\dagger_k = \psi^\dagger_{f_{N-k+1}+1}\ \psi_{f_{N-k+1}} = \Phi_{f_{N-k+1}+1, f_{N-k+1}}$. In this way, by composing together different σ_k operations we can move individual fermions around. Clearly, all the N σ_k operations are necessary in order to move each of the N fermions indvidually. It is easy to see that by applying sufficient number of such fermion bilinears one can move to any desired fermion state starting from a given state.

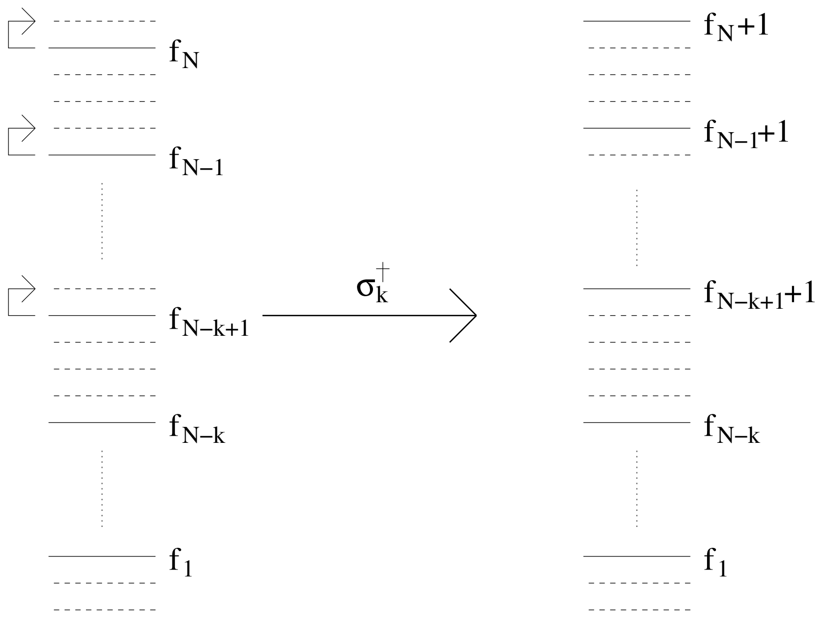

Figure 1: The action of $\sigma_k^\dagger$.

It follows from the definition of $\sigma_k^\dagger$, σ_k operators that they satisfy the following relations:

$$\sigma_k\ \sigma_k^\dagger = 1, \quad \sigma_k^\dagger\ \sigma_k = \theta(r_k - 1), \quad [\sigma_l, \sigma_k^\dagger] = 0, \quad l \neq k, \tag{2.5}$$

where $(f_{N-k+1} - f_{N-k} - 1) \equiv r_k$ and $\theta(m) = 1$ if $m \geq 0$, otherwise it vanishes. Moreover, all the σ_k's annihilate the Fermi vacuum.

Consider now a system of bosons each of which can occupy a state in an N-dimensional Hilbert space $\mathcal{H}_N$. Suppose we choose a basis $\{|k\rangle,\ k = 1, \cdots, N\}$ of $\mathcal{H}_N$. In the second quantized notation we introduce creation (annihilation) operators $a_k^\dagger$ (a_k) which create (destroy) particles in the state $|k\rangle$. These satisfy the commutation relations

$$[a_k, a_l^\dagger] = \delta_{kl}, \quad k, l = 1, \cdots, N \tag{2.6}$$

A state of this bosonic system is given by (a linear combination of)

$$|r_1, \cdots, r_N\rangle = \frac{(a_1^\dagger)^{r_1} \cdots (a_N^\dagger)^{r_N}}{\sqrt{r_1! \cdots r_N!}} |0\rangle \tag{2.7}$$

It can be easily verified that equations (2.5) are satisfied if we make the following identifications

$$\begin{aligned} \sigma_k &= \frac{1}{\sqrt{a_k^\dagger a_k + 1}} a_k, \\ \sigma_k^\dagger &= a_k^\dagger \frac{1}{\sqrt{a_k^\dagger a_k + 1}}, \end{aligned} \tag{2.8}$$

together with the map

$$\begin{aligned} r_k &= f_{N-k+1} - f_{N-k} - 1, \qquad k = 1,\, 2,\, \cdots N-1 \\ r_N &= f_1. \end{aligned} \tag{2.9}$$

This identification is consistent with the Fermi vacuum being the ground state of the bosonic system. The map (2.9) first appeared in [27]. The first of these arises from the identification (2.8) of σ_k's in terms of the oscillator modes, while the second follows from the fact that σ_N annihilates any state in which f_1 vanishes.

Using the above bosonization formulae, any fermion bilinear operator can be expressed in terms of the bosons. For example, the hamiltonian can be rewritten as follows. Let $\mathcal{E}(m),\ m = 0, 1, 2, \cdots$ be the exact single-particle spectrum of the fermions (assumed noninteracting). Then, the hamiltonian is given by

$$H = \sum_{m=0}^{\infty} \mathcal{E}(m)\ \psi_m^\dagger\ \psi_m. \tag{2.10}$$

Its eigenvalues are $E = \sum_{k=1}^{N} \mathcal{E}(f_k)$. Using $f_k = \sum_{i=N-k+1}^{N} r_i + k - 1$, which is easily derived from (2.9), these can be rewritten in terms of the bosonic occupation numbers, $E = \sum_{k=1}^{N} \mathcal{E}(\sum_{i=N-k+1}^{N} r_i + k - 1)$. These are the eigenvalues of the bosonic hamiltonian

$$H = \sum_{k=1}^{N} \mathcal{E}\left(\sum_{i=N-k+1}^{N} a_i^\dagger a_i + k - 1 \right). \tag{2.11}$$

This bosonic hamiltonian is, of course, completely equivalent to the fermionic hamiltonian we started with.

For the harmonic potential, the spectrum is linear. In this case we get

$$H - H_F = \sum_{k=1}^{N} k a_k^\dagger a_k, \tag{2.12}$$

where H_F is the Fermi ground state energy. This hamiltonian, and the LLM system that it describes, will be the focus of our discussions in the rest of this paper.

We remark that our bosonization technique does not depend on any specific choice of fermion hamiltonian and can be applied to various problems like $c = 1$, free fermions on a circle (the Tomonaga problem) [28] etc. Also see [1] for some more details on this issue.

3. Graviton interaction

In this section we will present a quantum computation of graviton correlators from the boundary theory. The main result of this computation, described in subsection 3.3, will be to show that for sufficiently high energy modes, the concept of gravitons breaks down since the strength of their interaction grows exponentially with N. We will show that such pathological behaviour can be understood as a wrong choice of variables to describe gravity at short wavelengths and the right variables to describe gravity at such energies

JHEP03(2006)031

are the giant gravitons in terms of which the interactions become weak. Another result of the computation will be to exhibit a chiral ring structure of the graviton interactions at low energies, which reduces multipoint graviton interactions to essentially a combination of the 'structure constants' of the chiral ring.

3.1 The exact correlators

We recall that the standard AdS/CFT dictionary identifies gravitons in the bulk to the single trace operators TrZ^m which are represented in the fermion theory by the operators[15]:

$$\beta^\dagger_m = \sum_{n=0}^{\infty} C(m,n)\psi^\dagger_{n+m}\psi_n, \tag{3.1}$$

$$C(m,n) \equiv \sqrt{\frac{(m+n)!}{2^m n!}} \tag{3.2}$$

We will denote correlators of the theory as

$$D(m_1,\ m_2,...,\ m_r|n_1,\ n_2,...,\ n_s) \equiv \langle F_0|\beta_{m_1}...\beta_{m_r}\beta^\dagger_{n_1}...\beta^\dagger_{n_s}|F_0\rangle \equiv \langle\beta_{m_1}...\beta_{m_r}\beta^\dagger_{n_1}...\beta^\dagger_{n_s}\rangle \tag{3.3}$$

Here $|F_0\rangle$ is the Fermi vacuum. An exact calculation of $D(m,n|m+n)$ and $D(m|m)$, valid for finite N, can be done using either the fermion representation for the β's (which is simpler) or their bosonic representation in terms of the oscillators $a_k,\ a^\dagger_k$ (see Appendix A). We quote the results below [5]:

$$\begin{aligned} D(m,n|m+n) = & \frac{1}{2^{m+n}(m+n+1)}\left[\frac{(N+m+n)!}{(N-1)!} + \frac{N!}{(N-m-n-1)!}\right. \\ & \left. - \frac{(N+m)!}{(N-n-1)!} - \frac{(N+n)!}{(N-m-1)!}\right], \end{aligned} \tag{3.4}$$

which is valid for $(m+n) < N$. For $(m+n) = N$, we get

$$D(m,N-m|N) = \frac{1}{2^N(N+1)}\left[\frac{(2N)!}{(N-1)!} - \frac{(N+m)!}{(m-1)!} - \frac{(2N-m)!}{(N-m-1)!}\right]. \tag{3.5}$$

Also,

$$D(m|m) = \frac{1}{2^m(m+1)}\left[\frac{(N+m)!}{(N-1)!} - \frac{N!}{(N-m-1)!}\right],\quad m < N, \tag{3.6}$$

and

$$D(N|N) = \frac{1}{2^N(N+1)}\frac{(2N)!}{(N-1)!}. \tag{3.7}$$

The normalized correlators are given by

$$\Gamma(m_1,...,m_r|n_1,...,n_s) = \frac{D(m_1,...,m_r|n_1,...,n_s)}{\sqrt{D(m_1|m_1)..D(m_r|m_r)D(n_1|n_1)..D(n_s|n_s)}}. \tag{3.8}$$

[5]Our results agree with calculations done earlier using matrix model [29].

We will mostly deal with $s = 1$, that is, correlators involving only several TrZ^{m_i} and one anti-holomorphic operator $Tr\bar{Z}^n$. These satisfy a non-renormalization theorem and one can consistently discuss them staying within the LLM sector (loop corrections, which typically involve intermediate states outside the LLM sector, are absent for these correlators). For $s = 1$, we have $n_s = n_1 = m_1 + ... + m_r$.

3.2 Graviton interaction at long wavelengths

Let us first consider interaction of gravitons at low energies, where the frequencies of the gravitons, m, n, will be considered to be small numbers held fixed in the large N limit.

We will begin by discussing the normalized correlators (3.8), and in particular the 3-point function for the first few modes $m, n = 1, 2, 3....$ Some examples, obtained using the exact formulae (3.4),(3.6) are

$$\begin{aligned} D(1|1) &= N/2,\ D(2|2) = N^2/2,\ D(3|3) = 3(N^3 + N)/8 \\ \Gamma(1,1|2) &= \frac{\sqrt{2}}{N} \end{aligned} \tag{3.9}$$

The general behaviour of the 3-point functions for various ranges of m, n is discussed in section 3.3. The fall-off with N of the three-point function in (3.9), namely $1/N$, is an example of the more general $\sqrt{mn(m+n)}/N$ behaviour mentioned there.

We also need the values of some normalized four-point functions, which can be calculated either using the formulae (3.2) or the relation between the β's and the $a, a^\dagger$ oscillators indicated in Appendix A. A simple example is

$$\Gamma(1,1,1|3) = \frac{2\sqrt{3}}{N^2}(1 + 1/N^2)^{-1/2} \tag{3.10}$$

3.2.1 Chiral ring structure

The three-point functions (3.4) imply following chiral ring relation:

$$\begin{aligned} \beta_n\, \beta_m &= \hat{C}_{mn}\beta_{m+n}[1 + O(1/N)] \\ \hat{C}_{mn} &\propto 1/N \end{aligned} \tag{3.11}$$

The 'structure constant' is given by

$$\frac{D(m,n|m+n)}{D(m+n|m+n)} = \hat{C}_{mn}[1 + O(1/N)] \tag{3.12}$$

which follows from computing the correlator of both sides of (3.11) with $\beta^\dagger_{m+n}$. It is easy to see (using the approximation methods of section 3.3) that for large N and fixed m, n, the leading behaviour of $\hat{C}_{mn}$ is $1/N$. We will provide evidence for (3.11) by proving the following 'bootstrap' property of the 4-point function.

3.2.2 Four-point function and 'bootstrap'

If the relation (3.11) is true, it follows that the four-point function must satisfy what we will call a "bootstrap relation" (see figure 2).

JHEP03(2006)031

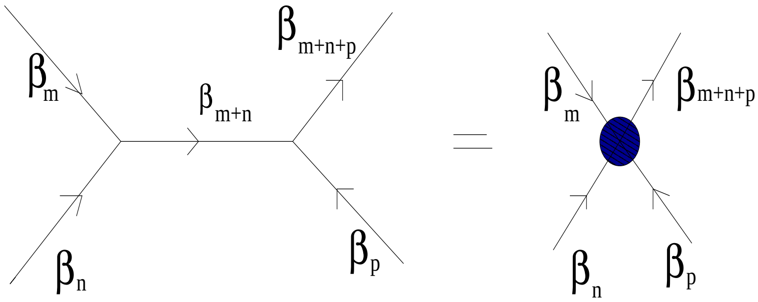

Figure 2: Bootstrap: Test for a cubic graviton field theory.

$$\begin{aligned}\langle\beta_m\beta_n\beta_p\beta^\dagger_{m+n+p}\rangle &= \hat{C}_{mn}\langle\beta_{m+n}\beta_p\beta^\dagger_{m+n+p}\rangle[1+O(1/N)]\\ &= \frac{D(m,n|m+n)D(m+n,p|m+n+p)}{D(m+n|m+n)}[1+O(1/N)].\end{aligned} \tag{3.13}$$

Example: For $m=n=p=1$,

$$\begin{aligned}&\text{LHS} = D(1,1,1|3) = 3N/4,\\ &\text{RHS} = D(1,1|2)\frac{1}{D(2|2)}D(2,1|3) = N/2\frac{1}{N^2/2}3N^2/4 = 3N/4.\end{aligned} \tag{3.14}$$

(In this case the $O(1/N)$ correction is absent.)

Let us discuss eq. (3.13) in a little more detail. On general grounds

$$\langle F_0|\beta_m\beta_n\beta_p\beta^\dagger_{m+n+p}|F_0\rangle = \sum_n \langle F_0|\beta_m\beta_n|n\rangle\langle n|\beta_p\beta^\dagger_{m+n+p}|F_0\rangle \tag{3.15}$$

where $\sum_n |n\rangle\langle n|$ represents a sum over all states in the theory (we can restrict, for these correlators, to states in the LLM sector, which belong to the Fermion Fock space). eq. (3.13) implies, to the leading order in 1/N, that the only intermediate state that contributes is $\beta^\dagger_{m+n}|F_0\rangle$. This indeed turns out to be correct, because of the rather remarkable relation which is easy to prove,

$$\beta_p\beta^\dagger_{m+n+p}|F_0\rangle = N^{p-1}\left[\left(1+O\left(\frac{1}{N}\right)\right)\frac{p(p+m+n)}{2^p}\beta^\dagger_{m+n}\Big|F_0\rangle + O\left(\frac{1}{N}\right)\beta^\dagger\beta^\dagger\cdots\Big|F_0\rangle\right] \tag{3.16}$$

where the last term is in general proportional to a multi-graviton state (denoted by multiple $\beta^\dagger$ acting on the vacuum). To leading order in $1/N$, this term is the 2-graviton state. We explicitly list a few examples:

$$\begin{aligned}\beta_1\beta^\dagger_3|F_0\rangle &= \frac{3}{2}\beta^\dagger_2|F_0\rangle\\ \beta_1\beta^\dagger_4|F_0\rangle &= 2\beta^\dagger_3|F_0\rangle\\ \beta_2\beta^\dagger_4|F_0\rangle &= N[2\beta^\dagger_2|F_0\rangle + \frac{1}{N}(\beta^\dagger_1)^2|F_0\rangle].\end{aligned} \tag{3.17}$$

JHEP03(2006)031

3.2.3 Cubic gravity theory

The above discussion about the chiral ring suggests a field theory of gravitons of the structure

$$S = \int D_m \beta_m^\dagger \beta_m + C_{mn} \beta_{m+n}^\dagger \beta_m \beta_n + c.c. \tag{3.18}$$

where the D_m and C_{mn} are related respectively to $D(m|m)$ and $\hat{C}_{mn}$, upto symmetry factors. The notations C_{mn}, $\hat{C}_{mn}$ and $C(m,n)$, defined respectively in (3.11), (3.18) and (3.2) are to be distinguished.

The cubic interactions in (3.18) can be derived from (3.25) in $1/N$ expansion. They arise solely from the definition of β's as linear combinations of multi-$a^\dagger$ oscillator states. The theory (3.18) has been matched to IIB supergravity in $AdS_5 \times S^5$ in the LLM sector in the work [31].

A suggestion similiar to the above has also been made recently in the work [30]. Evidence in support of this suggestion has been presented by matching direct computations of some correlation functions in the fermion theory with cubic field theory.

3.3 Breakdown of the graviton description

In this section we will study the behaviour of the normalized three-point function derived from (3.4)- (3.8). In the discussion below we will restrict ourselves to the case when all the three gravitons have energies much smaller than N. It turns out that there are three separate energy regimes of interest. Let us consider the three cases in turn.

1. m, n fixed as $N \to \infty$. This is the large-N perturbative regime. Here

$$\Gamma(m,\ n|m+n) = \frac{\sqrt{mn(m+n)}}{N} + O(1/N^2) \tag{3.19}$$

This result can be obtained by a straightforward $1/N$ expansion of (3.4)-(3.7) and it agrees with the calculation of tree-level three-graviton amplitude in supergravity [31].

2. $m/\sqrt{N} \sim n/\sqrt{N}$ fixed (and $O(1)$), as $N \to \infty$. Here

$$\Gamma(m,\ n|m+n) \approx a N^{-1/4} \tag{3.20}$$

The quantity a is a function of the fixed ratio $m^2/N, a = f(m^2/N)$. This result has been obtained from the formula

$$\Gamma(m,\ n|m+n) \approx \sqrt{\frac{m}{N}} \frac{\sinh^2 \frac{m^2}{N}}{\sinh \frac{m^2}{2N} \sqrt{\sinh \frac{2m^2}{N}}}, \tag{3.21}$$

which can be derived from (3.4)-(3.7) using the formulae given in Appendix B. Although the overall strength of the interaction, $\sim N^{-1/4}$, is small, to recover the function $f(m^2/N)$ (involving $\sinh(m^2/N)$ etc.) from a perturbation theory in $1/N$, e.g. supergravity, one needs to sum over all orders in $1/N$. We see that in this regime of energies, perturbation theory breaks down (in the sense that no finite order calculation in $1/N$ will reproduce the above result).

3. $m/N \sim n/N$ fixed and small as $N \to \infty$. Here

$$\Gamma(m,\ n|m+n) \approx a \exp[bN],\ \ a \propto \sqrt{m/N},\ \ b \propto m^2/N^2 \tag{3.22}$$

   In this case the exponent b turns out to be positive. Hence for m a small fixed fraction of N, the correlator grows exponentially with N! This means that gravitons become strongly interacting in this energy regime.

In fact, already for energies m which grow as $N^{1/2+\gamma}$, for $\gamma > 0$, the gravitons become strongly interacting and are not useful concepts as particles. What, then, are the new weakly interacting entities which can exist as perturbative states?

In the boundary theory that we are considering here, there are two possible candidates. One is to replace gravitons by the oscillator excitations of the bosonized theory discussed in section 2. These excitations are exactly non-interacting and hence they can replace gravitons at high energies. The corresponding bulk degrees of freedom, at the level of single-particle states (see sec 3.4.1), are the giant gravitons. The other possibility is to replace single-graviton states by the Schur polynomial combinations of multi-graviton states (which we will henceforth refer to merely as the "Schurs"). The main reason for this choice is that at high energies, Schurs are weakly interacting. This can be seen from a calculation of multi-point correlation functions of Schur poloynomials, $\chi_m(Z)$, which has been done in [13]. From this work we can read off the normalized 3-point function of the Schurs:

$$\tilde{\Gamma}(m,\ n|m+n) \equiv \frac{< \chi_m(Z)\chi_n(Z)\chi_{m+n}(\bar{Z}) >}{||\chi_m(Z)||\ ||\chi_n(Z)||\ ||\chi_{m+n}(\bar{Z})||} = \sqrt{\frac{N!/(N-m-n)!}{N!/(N-m)!\ N!/(N-n)!}} \tag{3.23}$$

We see that

1. for m, n fixed, as $N \to \infty$,

$$\tilde{\Gamma}(m,\ n|m+n) \sim O(1) \tag{3.24}$$

   Clearly the Schur polynomials do not represent weakly interacting particles for long wavelength modes. The gravitons (correspondig to single trace operators), with interactions $\sim O(1/N)$, are a better description of the low-energy perturbative spectrum. However,

2. for $m/N \sim n/N$ fixed and small as $N \to \infty$, we get

$$\tilde{\Gamma}(m,\ n|m+n) \sim e^{-aN}$$

   where a is a positive quantity of $O(1)$. We see that Schurs have exponentially small interactions in this regime of energies, unlike the gravitons whose interaction grows exponentially at such large energies.

   For $m/N \sim 1$ fixed and $n = 1$ [6], the 3-point function of Schurs [7] goes as

$$\tilde{\Gamma}(m, 1|m+1) = \sqrt{1 - m/N}\,.$$

[6] Here $n = 1$ has been taken for convenience. One could have taken it to be any number of order one, not necessarily exactly one.

[7] It would be more appropriate to think of this case as the coupling of a Schur to a graviton.

So even in this case there is a small but non-zero interaction.

The above observations teach us two things: (i) description of gravitons as perturbative spectrum breaks down for sufficiently large energies (which are, however, still $\ll N$), (ii) there is, nevertheless, a weakly coupled description available of the bulk physics, not in terms of the old gravitons, but in terms of (the bulk counterpart of) either the oscillator states of the bosonized theory or the Schurs. It turns out that the Schurs are closely related to the single-particle states created by the bosonic oscillators from the fermi vacuum, a fact that we will prove in section 3.5. Remarkably, therefore, both the choices lead to giant gravitons as the right choice of degrees of freedom to replace gravitons at high energies.

3.4 The universal bosonic excitations

It would appear from the above discussion of the three-point functions that there is a change of description of the perturbative spectrum from gravitons to giant gravitons as one tunes the energy up from low to sufficiently high. In principle, we could describe bulk physics at all energies in terms of bulk duals of the oscillators of the bosonized theory which create 'particle' states whose interaction strictly vanishes. In the boundary description these are the particle states created by the bosonic operators $a^\dagger_{m_1}, a^\dagger_{m_2}$... These have an energy cut-off $m = N$ by construction. The hamiltonian is exactly diagonal in terms of these oscillators

$$\begin{aligned} H &= \sum_{k=1}^{N} k a^\dagger_k a_k \\ &= \frac{1}{2}(N+1)\sum_{\mu=0}^{N-1} \phi^\dagger_\mu\ \phi_\mu + \frac{1}{2}\sum_{\mu\neq\nu=0}^{N-1}[1 - i\ \cot\frac{\pi}{N}(\mu-\nu)]\ \phi^\dagger_\mu\ \phi_\nu \end{aligned} \tag{3.25}$$

The second equality above is the "coordinate space" representation of the same hamiltonian, where ϕ_μ is a "lattice" Fourier transform of a_m [1].

This implies that there is a universal description of the perturbative spectrum in the half-BPS sector in terms of states which are non-interacting at all energies (with an in-built cut-off at $m = N$). Both the single trace operators and Schur polynomials create states which are linear combinations of these states.

3.4.1 Bulk interpretation of the states $a^\dagger_{m_1}a^\dagger_{m_2}...|0\rangle$

It is clear that the bulk map of the states $a^\dagger_{m_1}a^\dagger_{m_2}...|0\rangle$ is a linear combination of graviton states, as given by the equations (A.1)-(A.3) in Appendix A. Although the $a^\dagger$-particle states are free, the gravitons interact because of such linear combinations. E.g., using (A.2) and (A.3), we get

$$\begin{aligned} \langle F_0|\beta_1\beta_1\beta_2^\dagger|F_0\rangle &= \langle 0|\left(\frac{1}{2}\sqrt{N(N-1)}\sigma_2^\dagger + \frac{1}{2}\sqrt{N(N+1)}\left(\sigma_1^\dagger\right)^2\right)\times \\ &\quad \left(-\frac{1}{2}\sqrt{N(N-1)}\sigma_2^\dagger + \frac{1}{2}\sqrt{N(N+1)}(\sigma_1^\dagger)^2\right)|0\rangle \\ &= \frac{N}{2}\,. \end{aligned} \tag{3.26}$$

Normalizing according to (3.8), we recover $\Gamma(11|2)$ as given in (3.9). Note that $\langle 0|(a_1)^2) \times a_2^\dagger|0\rangle = 0$, as expected of strictly independent particle states.

As one considers higher and higher modes m, each β_m involves a larger number of multi-$a^\dagger$ states and the graviton interactions become stronger.

The states $a_{m_1}^\dagger a_{m_2}^\dagger ...|0\rangle$ have a closer relation to the states created at the boundary by the Schur polynomials. Indeed, as we will see in the next section 3.5, the single-particle states $a_n^\dagger|0\rangle$ are identical to the states created by the Schur polynomials for totally antisymmetric representations.

Multiple Schurs create states which are not identical to multi-$a^\dagger$ states (see, e.g. (3.32)). Since the giant gravitons do not appear to have perturbative open string excitations in the half-BPS sector [32], it is likely that the giant gravitons do not interact perturbatively. This behaviour is consistent with the multi-$a^\dagger$ states at the boundary, since these are completely non-interacting. Also, these boundary states have an inherent energy cut-off, consistent with the giant gravitons. Unlike these states, the Schur states are weakly interacting at high energies, while they interact strongly at low energies. It would seem from these considerations that it is the multi-$a^\dagger$ states which corresponds to multiple giant gravitons. However, this needs to be confirmed by direct calculations of giant graviton interactions in the bulk string/gravity theory.

3.5 Schurs and the bosonic oscillators

In this section, we will discuss a relation between the single-particle bosonic excitations and Schur polynomial excitations. From our discussion in section 2, explicit formulae for the graviton operators $\beta_m^\dagger$, eq. (3.2), acting on the fermi vacuum can be easily translated into bosonized formulae in terms of the bosonic oscillators $\sigma_i^\dagger$ acting on the vacuum state. Some examples of this have been given in Appendix A. These can then be inverted to express the latter in terms of the former.

We can get an idea of the meaning of the $\sigma_k^\dagger|0\rangle$ states by explicitly calculating them for a few small values of k in terms of multiple β's acting on the fermi vacuum. Taking appropriate linear combinations of $\beta_1^\dagger|F_0\rangle$, $\beta_2^\dagger|F_0\rangle$, $\beta_3^\dagger|F_0\rangle$, $\left(\beta_1^\dagger\right)^2|F_0\rangle$, $\left(\beta_1^\dagger\right)^3|F_0\rangle$ and $\beta_1^\dagger\beta_2^\dagger|F_0\rangle$, we find that

$$\begin{aligned} C(1,N-1)\sigma_1^\dagger|0\rangle &= \beta_1^\dagger|F_0\rangle, \\ C(2,N-2)\sigma_2^\dagger|0\rangle &= \frac{1}{2!}\left[\left(\beta_1^\dagger\right)^2 - \beta_2^\dagger\right]|F_0\rangle, \\ C(3,N-3)\sigma_3^\dagger|0\rangle &= \frac{1}{3!}\left[\left(\beta_1^\dagger\right)^3 - 3\beta_1^\dagger\beta_2^\dagger + 2\beta_3^\dagger\right]|F_0\rangle, \end{aligned} \tag{3.27}$$

where $C(m,n)$ are defined in (3.2). These can be generated from the formula

$$\sum_{m=1}^{k}(-1)^{k-m}\beta_m^\dagger\tilde{\sigma}_{k-m}^\dagger|0\rangle + (-1)^k k\tilde{\sigma}_k^\dagger|0\rangle = 0, \tag{3.28}$$

(for $k = 1,2,3$) where

$$\tilde{\sigma}_k^\dagger \equiv C(k,N-k)\sigma_k^\dagger. \tag{3.29}$$

Note that in writing the equations (3.27) and (3.28) we have implicitly used the fact that the Fermi vacuum, $|F_0\rangle$, and the bosonic vacuum, $|0\rangle$, are the same state. It thus makes sense to have the β's acting on either $|F_0\rangle$ or $|0\rangle$.

As is proven in Appendix C, the recursion relation (3.28) is actually valid at a general level $k = 1, 2, \ldots$, and so generates all the $\tilde{\sigma}_k^\dagger|0\rangle$'s in terms of multiple $\beta_k^\dagger$'s acting on the vacuum. This means that the single-particle states created by the bosonic oscillators $\sigma_k^\dagger$ acting on the vacuum are identical to the single-particle states created by the Schur combinations of single and multi-particle graviton states. This is because the operators on right-hand side of (3.27) have precisely the form of Schur polynomials in the completely antisymmetric representation. Denoting the Schurs by s_k, where $k = 1, 2, 3, \cdots$, we have

$$\begin{aligned} \mathrm{s}_1 &= \mathrm{Tr}X, \\ \mathrm{s}_2 &= \frac{1}{2}\left[(\mathrm{Tr}X)^2 - \mathrm{Tr}X^2\right], \\ \mathrm{s}_3 &= \frac{1}{6}\left[(\mathrm{Tr}X)^3 - 3\mathrm{Tr}X^2\mathrm{Tr}X + 2\mathrm{Tr}X^3\right], \end{aligned} \tag{3.30}$$

etc. where X is a hermitian matrix. These relations follow from the recursion formula [33]

$$\sum_{m=1}^{k}(-1)^{k-m}\mathrm{Tr}X^m s_{k-m} + (-1)^k k s_k = 0, \tag{3.31}$$

which defines all the Schur polynomials in the completely antisymmetric representation[8]. This equation can be proven using Newton's identitites [35, 36]. Comparing the recursion relation (3.28) with (3.31), we see that the expressions for Schur polynomials, s_k's, in terms of polynomials of the traces TrX^m of the matrix X are identical to the expressions for the oscillators $\tilde{\sigma}_k^\dagger$ in terms of polynomials of β_m's (acting on vacuum). In the following we will denote these polynomials of β_k's also as s_k's, which is justified because of the equivalence between the β_m's and single trace operators, TrX^m.

The multi-particle states $a_{m_1}^\dagger a_{m_2}^\dagger ...|0\rangle$ are, however, different from the states created by the Schur polynomials. For example, we have

$$\begin{aligned} (s_1^\dagger)^2|F_0\rangle &= (\beta_1^\dagger)^2|F_0\rangle \\ &= \frac{1}{2}\sqrt{N(N-1)}\sigma_2^\dagger|0\rangle + \frac{1}{2}\sqrt{N(N+1)}(\sigma_1^\dagger)^2|0\rangle. \end{aligned} \tag{3.32}$$

We see that the multi-Schur state is in general a linear combination of multi-particle states of the bosonic oscillators. Moreover, using the fact that

$$s_2^\dagger|F_0\rangle = \frac{1}{2}\sqrt{N(N-1)}\sigma_2^\dagger|0\rangle$$

[8]The Schur polynomials in the completely antisymmetric representation are also known as Chern polynomials [33, 34]. In [33], the latter are denoted by c_k, and appear in the context of topological classifications of manifolds. The matrix under consideration is formed from components of the Ricci 2-form $\mathcal{R}_{\alpha\beta\gamma\delta}dx^\gamma \wedge dx^\delta$. The corresponding Chern polynomials can be shown to be closed $2k$-forms, hence defining cohomology classes on the manifold.

we find, using the definition $||s_i|| \equiv ||s_i^\dagger|F_0\rangle||$,

$$\frac{\langle F_0|s_1^2 s_2^\dagger|F_0\rangle}{||s_1||^2\ ||s_2||} = \sqrt{1-1/N},$$

which is an example of the $O(1)$ interaction among the Schur polynomial states (see (3.24)). This shows once again that multi-Schur states are different from multi-particle states of the bosonic oscillators.

4. Summary and discussion

In this paper we have studied non-perturbative quantum dynamics of the LLM (half-BPS) fluctuations around $\mathrm{AdS}_5\times\mathrm{S}^5$ using its correspondence to the boundary super Yang-Mills theory. We have seen that a description of the bulk gravitational physics in terms of the perturbative graviton states breaks down at sufficiently high energies. This is expected in any theory of gravity. But in the example studied here, we can go further and identify the new weakly coupled degrees of freedom in terms of which the bulk physics must be described at high energies. We have argued that these are the giant graviton states. A remarkable thing about the LLM sector is that all the states in this sector, namely gravitons, giants, Schurs, etc. can be described in terms of the set of N free bosonic oscillators $a_k, a_k^\dagger$. From this point of view interactions emerge as a result of the fact that gravitons, giants, Schurs, etc. are linear combinations of multi-particle states created by these oscillators.

An interesting feature of gravity in the LLM sector is that perturbation theory remains valid until energies of order $N^{1/2}$ are reached. General arguments for validity of perturbative gravity would have given the relevant (dimensionless) scale to be $R/l_p \sim N^{1/4}$. Presumably the high degree of supersymmetry in the LLM sector is responsible for this, but it would be interesting to have an explicit argument. A related fact [22] is that while the size (in AdS_5) of a graviton excitations becomes smaller than the 10$-$dimensional Planck scale for energies larger than $N^{1/2}$, the size of giant gravitons on S^5 becomes larger than Planck scale for angular momenta larger than $N^{1/2}$. This also seems to suggest that a meaningful description of physics in the bulk can be constructed in terms of giant gravitons precisely at and beyond those energies where one might expect the graviton description to break down.

Another question that has arisen from the present investigation is about the identification of the boundary states corresponding to multi-giants. We have seen that at the level of single-particle states, Schurs in totally antisymmetric represenatations are identical to the states created by the oscillators $a_k^\dagger$. However, since muti-Schur states are different from multi-$a_k^\dagger$ states, the former being interacting while the latter are free, one might ask which of these correspond to the bulk multi-giant states. Clearly, this question can only be settled by computations of interactions of giants among themselves and with gravitons (all staying within the half-BPS sector) in the bulk string theory or its semiclassical gravity limit.

JHEP03(2006)031

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A. Beta's and oscillators

In this Appendix we will collect some useful formulae about the graviton operators, $\beta_k^\dagger$. Acting on the fermi vacuum and using the bosonization formulae, we can write a simple expression for $\beta_k^\dagger|F_0\rangle$ in terms of the bosonic oscillators. We get,

$$\begin{aligned}\beta_m^\dagger\,|F_0\rangle = & \sum_{n=2}^{N}(-1)^{n-1}\sqrt{\frac{(N+m-n)!}{2^m(N-n)!}}\,\theta(m-n)\,\sigma_1^{\dagger\,m-n}\,\sigma_n^\dagger\,|0\rangle \\ & + \sqrt{\frac{(N+m-1)!}{2^m(N-1)!}}\,\sigma_1^{\dagger\,m}\,|0\rangle.\end{aligned}\tag{A.1}$$

Note that the σ_k's are related to the oscillator a_k's by (2.8). Also, $\theta(m)=1$ if $m\geq 0$, otherwise it vanishes. We list below the first few of these explicitly:

$$\begin{aligned}\beta_1^\dagger|F_0\rangle &= \sqrt{\frac{N}{2}}\sigma_1^\dagger|0\rangle, \\ \beta_2^\dagger|F_0\rangle &= -\frac{1}{2}\sqrt{N(N-1)}\sigma_2^\dagger|0\rangle+\frac{1}{2}\sqrt{N(N+1)}(\sigma_1^\dagger)^2|0\rangle,\end{aligned}\tag{A.2}$$

etc. The action of multiple β's on fermi vacuum can also be expressed in terms of the bosonic oscillators. For example,

$$(\beta_1^\dagger)^2|F_0\rangle = \frac{1}{2}\sqrt{N(N-1)}\sigma_2^\dagger|0\rangle+\frac{1}{2}\sqrt{N(N+1)}(\sigma_1^\dagger)^2|0\rangle,\tag{A.3}$$

etc. This requires a more general bosonization formula than (A.1), which can be easily obtained using the discussion in section 2. Finally, we note that beyond $m=N$, there is no single-particle piece on the right hand side. For example,

$$\begin{aligned}\beta_{N+1}^\dagger\,|F_0\rangle = & \sum_{n=2}^{N}(-1)^{n-1}\sqrt{\frac{(N+m-n)!}{2^m(N-n)!}}\,\sigma_1^{\dagger\,N+1-n}\,\sigma_n^\dagger\,|0\rangle \\ & + \sqrt{\frac{(2N)!}{2^{N+1}(N-1)!}}\,\sigma_1^{\dagger\,N+1}\,|0\rangle,\end{aligned}\tag{A.4}$$

which is a linear combination of multi-particle states, with no single-particle component.

B. Derivation of eqn. (3.21)

We begin by noting the useful formula:

$$\frac{(N+p_1)!}{(N-p_0-1)!} = N^{p_0+p_1+1}\exp\left[\sum_{k=0}^{p_0+p_1}\ln\left(1+\frac{p_1-k}{N}\right)\right]$$

$$= N^{p_0+p_1+1} \exp\left[\frac{1}{2N}(p_1 - p_0)(p_0 + p_1 + 1) + O\left(\frac{1}{N^2}\right)\right] \quad \text{(B.1)}$$

where the first line is exact and the second line is valid for $p_0/N, p_1/N \ll 1$. The $O(1/N^2)$ term actually evaluates to $-\frac{1}{6N^2}(p_1^2 + p_0^2 - p_0 p_1 + \frac{p_0+p_1}{2})(p_0 + p_1 + 1)$.

Using this approximation repeatedly, it is easy to prove

$$\begin{aligned} \langle \beta_m \beta_m^\dagger \rangle &\approx 2T(m) \sinh \frac{m(m+1)}{2N}, \\ \langle \beta_m \beta_n \beta_{m+n}^\dagger \rangle &\approx 4T(m+n) \sinh\left(\frac{m(m+n+1)}{2N}\right) \sinh\left(\frac{n(m+n+1)}{2N}\right), \end{aligned} \quad \text{(B.2)}$$

where

$$T(m) \equiv \frac{N^{m+1}}{2^m(m+1)}. \quad \text{(B.3)}$$

For $m \sim n$, the neglected terms (in the arguments of the sinh) are of order m^3/N^2. Compared to the leading term ($\sim m^2/N$), this is down by a factor of m/N. These can, therefore, be neglected if we assume $m/N \ll 1$. We can now use the two equations above to arrive at (3.21).

C. Proof of eqn. (3.28)

In this section, we will prove equation (3.28). We will utilize the intuition derived from the bosonization picture. The fundamental object of interest to us is $\beta_m^\dagger \sigma_{k-m}^\dagger |0\rangle$. First, the bosonic creation operator $\sigma_{k-m}^\dagger$ lifts the $(k-m)$ top fermions by one step, creating a hole at level $N - k + m$. The action of the graviton operator $\beta_m^\dagger$, (3.2), on this state is to lift a fermion in any one of the occupied levels by another m steps, which can happen in three qualitatively different ways, as shown in figure 3. In the first case, indicated by the lower right arrow in figure 3, one trades a hole for another in the lower heap of occupied levels. Alternatively, one may place the fermion from the lower heap (which requires the level n of the fermion annihilation operator, see (3.2), to satisfy $n \leq (N - k + m - 1)$) on top of everything (which requires the level $(m + n)$ of the fermion creation operator to satisfy $(m+n) \geq (N+1)$). This case is indicated by lower left arrow in figure 3. In this case, the topmost "chunk" consists only of a single fermion. Finally, if the fermion originated from the top heap, as indicated by the top right arrow in figure 3, we must satisfy the conditions $n \geq (N - k + m + 1)$ and $(m + n) \geq (N + 1)$.

The final state obtained in this way can be described in terms of bosonic creation operators acting on the vacuum. We need to be careful about relative signs arising from moving the annihilation operator ψ_n of equation (3.2) down by m steps as compared to the position of the $\psi_{n+m}^\dagger$, passing through chunks of fermions on its way. Taking all this

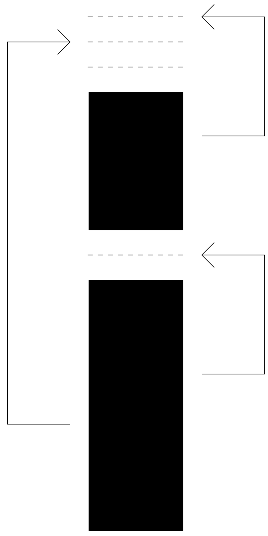

Figure 3: Forming the state $\beta^\dagger_m \sigma^\dagger_{k-m}|0\rangle$.

into account, the result turns out to be

$$\begin{aligned}
\beta^\dagger_m \sigma^\dagger_{k-m}|0\rangle &= (-1)^{m-1} C(m, N-k)\sigma^\dagger_k|0\rangle \\
&+ \sum_{n=N-m+1\ (m\le k/2)}^{n=N} (-1)^{N-n} C(m,n)\sigma^\dagger_{k-m}\sigma^\dagger_{N-n+1}(\sigma^\dagger_1)^{n+m-N-1}|0\rangle \\
&+ \sum_{n=N-k+m+1\ (m>k/2)}^{n=N} (-1)^{N-n} C(m,n)\sigma^\dagger_{k-m}\sigma^\dagger_{N-n+1}(\sigma^\dagger_1)^{n+m-N-1}|0\rangle \\
&+ \sum_{n=N-m+1\ (m>k/2)}^{n=N-k+m-1} (-1)^{N-n-1} C(m,n)\sigma^\dagger_{N-n}\sigma^\dagger_{k-m+1}(\sigma^\dagger_1)^{n+m-N-1}|0\rangle,
\end{aligned} \tag{C.1}$$

where the three sums only contribute for $m \le k/2$, $m > k/2$ and $m > k/2$, respectively, as indicated. Here and in the following we have assumed that k is even. The odd k case can be handled similarly.

In principle, we should now use the expression (C.1) to prove that (3.28) holds at a general level k. However, (3.28) can actually be split into two parts which cancel independently of each other. The first part is

$$\begin{aligned}
&\sum_{m=1}^{k}(-1)^{k-m}(-1)^{m-1}C(m,N-k)C(k-m,N+m-k)\sigma^\dagger_k|0\rangle \\
&\qquad\qquad +(-1)^k kC(k,N-k)\sigma^\dagger_k|0\rangle = \\
&\qquad = (1-1)(-1)^{k-1}kC(k,N-k)\sigma^\dagger_k|0\rangle = 0.
\end{aligned} \tag{C.2}$$

where $C(m,N-k)C(k-m,N+m-k) = C(k,N-k)$ was used. Using (C.1) and (C.2), what then remains to prove (3.28) is that

$$0=\sum_{1\le m\le k/2}(-1)^{k-m}\sum_{n=N-m+1}^{n=N}(-1)^{N-n}\alpha_{mn}\sigma^\dagger_{k-m}\sigma^\dagger_{N-n+1}(\sigma^\dagger_1)^{n+m-N-1}|0\rangle$$
$$+\sum_{k/2<m\le k}(-1)^{k-m}\sum_{n=N-k+m+1}^{n=N}(-1)^{N-n}\alpha_{mn}\sigma^\dagger_{k-m}\sigma^\dagger_{N-n+1}(\sigma^\dagger_1)^{n+m-N-1}|0\rangle \tag{C.3}$$
$$+\sum_{k/2<m\le k}(-1)^{k-m}\sum_{n=N-m+1}^{n=N-k+m-1}(-1)^{N-n-1}\alpha_{mn}\sigma^\dagger_{N-n}\sigma^\dagger_{k-m+1}(\sigma^\dagger_1)^{n+m-N-1}|0\rangle,$$

where

$$\alpha_{mn}\equiv C(m,n)C(k-m,N+m-k). \tag{C.4}$$

To complete the proof, we proceed as follows. To make summands more equal, shift $n\to n-1$ and $m\to m+1$ in the last sum over the index m. Redefining

$$\begin{aligned} n&\equiv N-q+1\\ m&\equiv k-p \end{aligned} \tag{C.5}$$

and dividing by $(-1)^k$ then turns (C.3) into

$$0=\sum_{p=k/2}^{k-1}\sum_{q=1}^{k-p}\gamma_{pq}\sigma_{pq}+\sum_{p=1}^{k/2-1}\sum_{q=1}^{p}\gamma_{pq}\sigma_{pq}-\sum_{p=1}^{k/2}\sum_{q=p}^{k-p}\tilde{\gamma}_{pq}\sigma_{pq}, \tag{C.6}$$

where γ_{pq} really just is shorthand for α_{mn} in the new indices,

$$\gamma_{pq}\equiv\alpha_{m=N-q+1,\ n=k-p}=C(p,N-p)C(k-p,N-q+1). \tag{C.7}$$

Furthermore, $\tilde{\gamma}_{pq}$ is the analogous coefficient in the third sum over the index m, in which the $m\to m+1$,$n\to n+1$ shifts were performed, i.e.

$$\begin{aligned}\tilde{\gamma}_{pq}&\equiv C(k-m-1,N+m-k+1)C(m+1,n-1)=\\ &=C(p-1,N-p+1)C(k-p+1,N-q).\end{aligned} \tag{C.8}$$

The virtue of (C.6) is that all the sums are now written in terms of the fundamental variable

$$\sigma_{pq}\equiv(-1)^{p+q}\sigma^\dagger_p\sigma^\dagger_q\left(\sigma^\dagger_1\right)^{k-(p+q)}|0\rangle. \tag{C.9}$$

Changing the order in which the terms in the last sum over the index p are summed using

$$\sum_{p=1}^{k/2}\sum_{q=p}^{k-p}=\sum_{q=1}^{k/2-1}\sum_{p=1}^{q}+\sum_{q=k/2}^{k-1}\sum_{p=1}^{k-q}, \tag{C.10}$$

swapping p for q in that sum, and using the facts that $\tilde{\gamma}_{qp} = \gamma_{pq}$ and $\sigma_{qp} = \sigma_{pq}$ finally turns the equation (C.6) that we want to prove into

$$\begin{aligned} 0 = & \sum_{p=k/2}^{k-1} \sum_{q=1}^{k-p} \gamma_{pq}\sigma_{pq} + \sum_{p=1}^{k/2-1} \sum_{q=1}^{p} \gamma_{pq}\sigma_{pq} \\ & - \sum_{p=1}^{k/2-1} \sum_{q=1}^{p} \gamma_{pq}\sigma_{pq} - \sum_{p=k/2}^{k-1} \sum_{q=1}^{k-p} \gamma_{pq}\sigma_{pq}, \end{aligned} \tag{C.11}$$

which is trivially true, hence concluding the proof of (3.28).

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JHEP03(2006)031